Discrete Mathematics: Combinatorics and Graph Theory

Homework 4: Due 12/17

Instructions. Solve any 10 questions. Typeset or write neatly and show your work to receive full credit.

1. Give the sample space (the set of possible outcomes) for the following:
   (a) Choosing a prime number less than 30 uniformly at random.
   (b) Choosing one letter uniformly at random from the word PROBABILITY.
   (c) Repeatedly rolling a die until getting a 6.
   (d) Give the size of the sample space of a five card poker hand.

2. Suppose that 5 percent of men and 0.25 percent of women are colorblind. If a colorblind person is chosen at random, what is the probability of that person being male (assuming that there are an equal number of males and females)? What if the population had twice as many males as females?

3. In a famous game show on television, a prize is placed behind one of three doors, with probability 1/3 for each door. The contestant chooses a door. The host then opens one of the other doors and says "As you can see, the prize is not behind this door. Do you want to stay with your original guess or switch to the remaining door?" When the contestant has chosen a wrong door, the host opens the other wrong door. When the contestant has chosen the right door, the host opens one of the two wrong doors, each with probability 1/2. Show that the contestant should switch.

4. Show that the expectation and variance of a Poisson random variable are both equal to λ.

5. Consider n envelopes with amounts $a_1, \cdots, a_n$ in dollars, where $a_1 \leq \cdots \leq a_n$. A gambler is presented two successive envelopes, with the probability being p_i that the envelopes contain a_i and a_{i+1} dollars, for $1 \leq i \leq n-1$. She opens one of these two envelopes at random and sees what it contains. She then can either keep that amount or switch to the other envelope. Suppose that she sees a_k dollars. In terms of the data of the problem, determine whether she should switch.

6. Recall that independent trials that result in a success with probability p and failure $(1-p)$ are called Bernoulli trials. Let P_n be the probability that n Bernoulli trials result in an even number of successes (0 being an even number). Show that:

$$P_n = p(1 - P_{n-1}) + (1-p)P_{n-1} \qquad \text{for } n \geq 1$$

   and use this formula to prove (by induction) that

$$P_n = \frac{1 + (1-2p)^n}{2}$$

7. Show that the Poisson random variable can be used to approximate a Binomial random variable when n is large and p is small. Hint: If X is a Binomial random variable with parameters (n, p) and the Poisson parameter $\lambda = np$, you want to show:

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} = e^{-\lambda} \frac{\lambda^x}{x!}$$

8. Let an denote the number of lattice paths of length $2n$ that never step above the diagonal (these end at some point $(k, 2n-k)$ with $k \geq n$). Prove that $a_n = \binom{2n}{n}$.

9. Consider a black box that when activated returns two Bernoulli trials whose outcomes are each H or T. That is, each time you run the procedure you return HH, HT, TH or TT. Can you design an algorithm that takes as input two possibly biased Bernoulli trials and returns a fair Bernoulli trial? You are allowed to discard certain events or simply rerun the black box instantly. Prove that your algorithm returns a single unbiased random variable.

10. Suppose you and a friend play a series of chess matches. Each game is independently won by you with a probability p and by your friend with a probability $1 - p$. You stop when the total number of wins of one person is two greater than that of the other. The individual with the greater number of total wins is declared the winner of the series.

    (a) Find the probability that a total of 4 games are played.

    (b) Find the probability that you are the winner of the series.

11. Gamma and Beta functions play an important role in defining probability densities of the same names and are natural analogues to functions and discrete distributions we have seen.

    (a) The Gamma function defined below can be used to interpolate the factorial.

    $$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$$

    Prove by induction that $\Gamma(x) = (x-1)!$ What discrete distribution is the product of an exponential and a polynomial (divided by a factorial)?

    (b) Perform a change of variables to show that $\Gamma(1/2) = \sqrt{\pi}$.

    (c) The Gamma function is also used to construct the Beta function and distribution.

    $$B(x, y) = \int_0^1 p^{x-1}(1-p)^{y-1} dp \qquad x, y > 0$$

    Comment on similarities between the Beta and Binomial distributions. Show that the distribution is normalized so that

    $$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$